進佳數學團隊 – Dr. Herbert Lam 林康榮博士

HKAL Pure Mathematics

F. Inequalities

1. Basic properties

Theorem 1

Let a, b, c be real numbers.

(i) If $a > b$ and $b > c$, then $a > c$.

(ii) If $a > b$ and $c > 0$, then $ac > bc$, but if $a > b$ and $c < 0$, then $ac < bc$.

Theorem 2

Let a and b be positive numbers.

Then $a > b$ if and only if $a^2 > b^2$.

The proof of above is simple and is left to the student.

Example 1

(a) Prove the inequality $\sqrt{n+1} - \sqrt{n} < \dfrac{1}{2\sqrt{n}} < \sqrt{n} - \sqrt{n-1}$, where $n > 0$.

(b) Hence show that $18 < 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{100}} < 19$.

(c) Show that $2\sqrt{n} - \dfrac{3}{2}$ is close to $S_n = 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}}$ if n is sufficiently large.

[Solution]

(a)
$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$
$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

$\because\ \sqrt{n+1} > \sqrt{n}$, i.e. $\sqrt{n+1} + \sqrt{n} > 2\sqrt{n}$,

$\therefore\ \dfrac{1}{\sqrt{n+1}+\sqrt{n}} < \dfrac{1}{2\sqrt{n}}$.

Hence $\sqrt{n+1} - \sqrt{n} < \dfrac{1}{2\sqrt{n}}$.

Similarly, we can show

$$\sqrt{n} - \sqrt{n-1} = \frac{1}{\sqrt{n}+\sqrt{n-1}}$$

$$\sqrt{n} + \sqrt{n-1} < 2\sqrt{n} \quad \Rightarrow \quad \frac{1}{\sqrt{n}+\sqrt{n-1}} > \frac{1}{2\sqrt{n}},$$

$$\frac{1}{2\sqrt{n}} < \sqrt{n} - \sqrt{n-1}.$$

(b) For $n=1$, we have $\sqrt{2}-1 < \dfrac{1}{2} < 1$.

But this inequality can be modified as

$$\sqrt{2}-1 < \frac{1}{2} \le 1 - \frac{1}{2} \cdots\cdots (*)$$

For $n = 2,3,\cdots,100$, we have

$$\sqrt{3}-\sqrt{2} < \frac{1}{2\sqrt{2}} < \sqrt{2}-1$$

$$\sqrt{4}-\sqrt{3} < \frac{1}{2\sqrt{3}} < \sqrt{3}-\sqrt{2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\sqrt{101}-\sqrt{100} < \frac{1}{2\sqrt{100}} < \sqrt{100}-\sqrt{99}$$

Adding up (*) and the others corresponding to $n = 2,3,\cdots,100$, we obtain

$$\sqrt{101}-1 < \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{100}}\right) < \sqrt{100}-\frac{1}{2}.$$

As $\sqrt{101}-1 > \sqrt{100}-1 = 9$,

therefore we have

$$9 < \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{100}}\right) < 10-\frac{1}{2}$$

i.e. $$18 < 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{100}} < 19.$$

(c) Similar to what we did in (b), consider the inequalities

$$\sqrt{2}-1<\frac{1}{2}\leq 1-\frac{1}{2}$$

$$\sqrt{3}-\sqrt{2}<\frac{1}{2\sqrt{2}}<\sqrt{2}-1$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\sqrt{n+1}-\sqrt{n}<\frac{1}{2\sqrt{n}}<\sqrt{n}-\sqrt{n-1}.$$

Adding up these, we have

$$\sqrt{n+1}-1<\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}\right)<\sqrt{n}-\frac{1}{2}$$

$$2(\sqrt{n+1}-1)<S_n<2(\sqrt{n}-\frac{1}{2}).$$

$\because$ $\sqrt{n+1}>\sqrt{n}$, $\therefore$ We obtain $S_n>2(\sqrt{n+1}-1)>2(\sqrt{n}-1)$.

Hence, $2\sqrt{n}-2<S_n<2\sqrt{n}-1$

$$-2<S_n-2\sqrt{n}<-1$$

$$-\frac{1}{2}<S_n-2\sqrt{n}+\frac{3}{2}<\frac{1}{2}$$

$$\left|S_n-\left(2\sqrt{n}-\frac{3}{2}\right)\right|<\frac{1}{2}.$$

This inequality implies that S_n is close to $2\sqrt{n}-\frac{3}{2}$ as n becomes large. For example, when $n=1,000,000$, S_n deviates from 1998.5 in less than $\frac{1}{2}$.

2. Well known inequalities

The triangle inequality, Cauchy-Schwarz's inequality and the inequality concerning arithmetic mean and geometric mean are presented in the following examples.

Example 2

Show that $|a+b|\leq|a|+|b|$.

[Solution]

Starting with $ab \le |ab|$,

we have $a^2 + 2ab + b^2 \le a^2 + 2|ab| + b^2$.

Since $a^2 = |a|^2$, $b^2 = |b|^2$, $|ab| = |a| \cdot |b|$,

the above inequality can be written as

$$a^2 + 2ab + b^2 \le |a|^2 + 2|a| \cdot |b| + |b|^2$$

$$(a+b)^2 \le (|a| + |b|)^2 .$$

$$\therefore \quad |a+b| \le |a| + |b| .$$

Equality holds if $a = b$.

Note: This is known as the **triangle inequality**, and is also valid when a and b are complex numbers or vectors. This will also be discussed in Chapter 4. The geometric interpretations for these two cases are the same, stated as follows.

The sum of any two sides of a triangle is longer than the third one, e.g. $PQ + QR > PR$ in Figure 1.

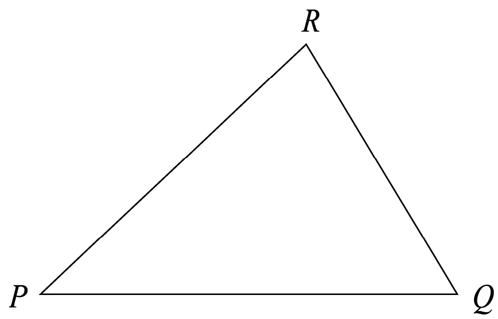

Figure 1

Example 3

Show that $(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \le (a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2) \cdot (b_1{}^2 + b_2{}^2 + \cdots + b_n{}^2)$,

where a_i's and b_i's are real numbers.

This is known as the **Cauchy-Schwarz's inequality**.

[Solution]

Consider $a_i x + b_i$, $i = 1, 2, \cdots, n$.

We have $\sum_{i=1}^{n} (a_i x + b_i)^2 \geq 0$, for all $x \in \mathbb{R}$.

$$\sum_{i=1}^{n} (a_i^2 x^2 + 2a_i b_i x + b_i^2) \geq 0$$

$$\therefore \left(\sum_{i=1}^{n} a_i^2 \right) x^2 + 2\left(\sum_{i=1}^{n} a_i b_i \right) x + \sum_{i=1}^{n} b_i^2 \geq 0.$$

Let $A = \sum_{i=1}^{n} a_i^2$, $B = \sum_{i=1}^{n} a_i b_i$, $C = \sum_{i=1}^{n} b_i^2$.

Then $Ax^2 + 2Bx + C \geq 0$.

The required inequality is trivial when $A = 0$. Suppose $A > 0$.

For the quadratic expression to be non-negative

$$\Delta = (2B)^2 - 4A \cdot C \leq 0$$

$$B^2 \leq AC$$

i.e. $$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 \right)\left(\sum_{i=1}^{n} b_i^2 \right).$$

Geometrically, the graph of $Ax^2 + 2Bx + C$ is above the x-axis .

The equality holds iff $\sum_{i=1}^{n} (a_i x + b_i)^2 = 0$ for some nonzero $x \in \mathbb{R}$.

Thus $a_i x + b_i = 0$, $i = 1, 2, \cdots, n$, i.e. $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n} \left(= -\frac{1}{x} \right)$.

Example 4

Given n positive numbers $a_1, a_2, \cdots, a_n$, show that the arithmetic mean (A.M.) is greater than or equal to the geometric mean (G.M.), i.e.

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

[Solution]

Proof of this well known inequality requires using the property of the natural logarithmic function, $\ln(x_1 x_2 \cdots x_n) = \ln x_1 + \ln x_2 + \cdots + \ln x_n$, where x_i's are positive.

First, we have to show when $x > 0$,

$\ln x \leq x - 1 \cdots\cdots (*)$

From Figure 2, we observe that the curve of $y = \ln x$ is below the line $y = x - 1$, and they touch each other at $x = 1$.

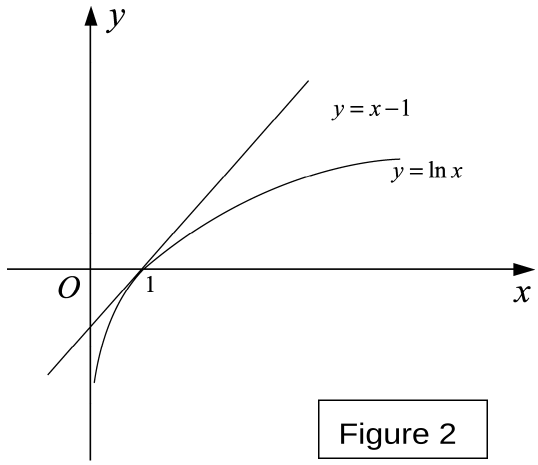

Figure 2

On the other hand, we can consider the function $f(x) = \ln x - x + 1$.

Since $f'(x) = \frac{1}{x} - 1 = 0 \Rightarrow x = 1$,

and $f''(x) = -\frac{1}{x^2} \Rightarrow f''(x) \leq 0$,

the curve of $f(x)$ attains a local maximum at $x = 1$ and the maximum value is zero. Hence, when $x > 0$, $f(x) \leq 0$, i.e. $\ln x \leq x - 1$.

Second, let $A = \frac{a_1 + a_2 + \cdots + a_n}{n}$, a_i's > 0.

By (*), we have

$$\ln\left(\frac{a_1}{A}\right) \leq \frac{a_1}{A} - 1,$$

$$\ln\left(\frac{a_2}{A}\right) \leq \frac{a_2}{A} - 1,$$

$$\vdots \qquad \vdots$$

$$\ln\left(\frac{a_n}{A}\right) \leq \frac{a_n}{A} - 1.$$

Adding up above,

$$\ln\left(\frac{a_1}{A}\right) + \ln\left(\frac{a_2}{A}\right) + \cdots \ln\left(\frac{a_n}{A}\right) \leq \frac{a_1}{A} + \frac{a_2}{A} + \cdots + \frac{a_n}{A} - n$$

$$\ln\left(\frac{a_1 a_2 \cdots a_n}{A^n}\right) \leq \frac{a_1 + a_2 + \cdots + a_n}{A} - n.$$

Note that RHS of the inequality $= n - n = 0$,

i.e. $\ln\left(\frac{a_1 a_2 \cdots a_n}{A^n}\right) \leq 0 \Rightarrow \frac{a_1 a_2 \cdots a_n}{A^n} \leq 1$

$$a_1a_2\cdots a_n \le \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^n$$

$$\frac{a_1+a_2+\cdots+a_n}{n} \ge \sqrt[n]{a_1a_2\cdots a_n}\ .$$

Equality holds if $a_1 = a_2 = \cdots = a_n$.

An alternative method of proving this inequality is to use the following inequality (**).

"If $x_1x_2\cdots x_n = 1$, then $x_1 + x_2 + \cdots + x_n \ge n$. $\cdots$ (**)"

Let $x_i = \dfrac{a_i}{\sqrt[n]{a_1a_2\cdots a_n}}$ and we have $\prod_{i=1}^{n} x_i = 1$.

Then $\sum_{i=1}^{n} x_i \ge n \quad \Rightarrow \quad \dfrac{a_1+a_2+\cdots+a_n}{\sqrt[n]{a_1a_2\cdots a_n}} \ge n$,

$$\frac{a_1+a_2+\cdots+a_n}{n} \ge \sqrt[n]{a_1a_2\cdots a_n}\ .$$

Note:

(i) (**) can be proved by induction on n.

(See Example 4 of Section A, Chapter 3, on P.107)

Otherwise, we can simply duplicate the above proof using the natural log function.

$\because \ \ln x_i \le x_i - 1$,

$\therefore \ \sum_{i=1}^{n} \ln x_i \le \sum_{i=1}^{n} (x_i - 1) \Rightarrow \ \ln \prod_{i=1}^{n} x_i \le \sum_{i=1}^{n} x_i - n$.

Since $x_1x_2\cdots x_n = 1$, LHS $= 0$.

Hence $\sum_{i=1}^{n} x_i \ge n$.

(ii) A.M. $\ge$ G.M. is frequently used to prove inequalities, and is considered more efficient than using algebraic manipulations in certain occasions. (see Example 6 on P.92)

Example 5

(a) Show that the sum of a positive number and its reciprocal is always greater than or equal to 2, i.e. $x+\frac{1}{x}\ge 2$ for $x>0$.

(b) Let p and q be positive numbers.
Show that $p^2q^2+p^2+q^2+1\ge 4pq$.

[Solution]

(a) It immediately follows from A.M. $\ge$ G.M. as

$$\frac{x+\frac{1}{x}}{2}\ge\sqrt{x\cdot\frac{1}{x}} \quad\Rightarrow\quad x+\frac{1}{x}\ge 2.$$

Alternatively, $x+\frac{1}{x}-2=\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2\ge 0$, $\quad\therefore\quad x+\frac{1}{x}\ge 2$.

(b) It is equivalent to show $\frac{p^2q^2+p^2+q^2+1}{pq}\ge 4$.

By (a), $pq+\frac{1}{pq}\ge 2$ and $\frac{p}{q}+\frac{q}{p}\ge 2$

$\therefore\quad pq+\frac{1}{pq}+\frac{p}{q}+\frac{q}{p}\ge 4$, $\qquad$ i.e. $\frac{p^2q^2+p^2+q^2+1}{pq}\ge 4$.

Example 6

Let a, b, c be distinct positive numbers.

(a) Show that $a^3+b^3>a^2b+ab^2$.

(b) Show that $a^2b+b^2a+b^2c+c^2b+c^2a+a^2c>6abc$.

(c) Hence, show that $a^3+b^3+c^3>3abc$.

[Solution]

(a) $(a+b)(a-b)^2>0 \quad (\because\ a+b>0)$

$a^3+b^3-a^2b-ab^2>0$, $\quad\therefore\quad a^3+b^3>a^2b+ab^2$.

(b) $(a-b)^2 > 0$,

$a^2+b^2 > 2ab$,

$a^2c+b^2c > 2abc$.

Similarly $b^2a+c^2a > 2abc$,

$c^2b+a^2b > 2abc$.

$\therefore \quad a^2b+b^2a+b^2c+c^2b+c^2a+a^2c > 6abc$.

(c) From (a) $a^3+b^3 > a^2b+ab^2$.

Similarly, $b^3+c^3 > b^2c+bc^2$,

$c^3+a^3 > c^2a+ca^2$.

Then $2(a^3+b^3+c^3) > a^2b+ab^2+b^2c+bc^2+c^2a+ca^2$.

From (b) $a^2b+b^2a+b^2c+c^2b+c^2a+a^2c > 6abc$.

This implies $2(a^3+b^3+c^3) > 6abc$,

$a^3+b^3+c^3 > 3abc$.

Note: This inequality can be proved in a different way. Since a, b and c are distinct positive numbers, so are a^3, b^3 and c^3.

Then $\dfrac{a^3+b^3+c^3}{3} > \sqrt[3]{a^3b^3c^3}$ (A.M.>G.M.)

implies $a^3+b^3+c^3 > 3abc$.

Example 7

Let a, b and c be positive numbers.

Using A.M.$\geq$G.M., show that $(1+a)(1+b)(1+c) \geq (1+\sqrt[3]{abc})^3$.

Under what condition on a, b and c will the equality hold?

[Solution]

Using A.M.$\geq$G.M.,

$\dfrac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$ and $\dfrac{1}{3}(ab+bc+ca) \geq \sqrt[3]{(ab)(bc)(ca)} = \sqrt[3]{(abc)^2}$.

Hence, $(1+a)(1+b)(1+c)-(1+\sqrt[3]{abc})^3$

$$=[1+(a+b+c)+(ab+bc+ca)+abc]-\left[1+3\sqrt[3]{abc}+3\sqrt[3]{(abc)^2}+abc\right]$$

$$=\left[(a+b+c)-3\sqrt[3]{abc}\right]+\left[(ab+bc+ca)-3\sqrt[3]{(ab)(bc)(ca)}\right]\geq 0$$

Equality holds if and only if

$\frac{1}{3}(a+b+c)=\sqrt[3]{abc}$ and $\frac{1}{3}(ab+bc+ca)=\sqrt[3]{(ab)(bc)(ca)}$

$\therefore$ $a=b=c$ and $ab=bc=ca$, i.e. $a=b=c$.

3. Inequalities involving polynomials

(a) Roots of a polynomial

Suppose we are able to find the roots of the polynomial equation $f(x)=0$ ...... (*).

Then $f(x)<0$ & $f(x)>0$ can be solved immediately. Consider the graph of $f(x)$ shown in Figure 3. It can be seen when $x_1\leq x\leq x_2$ or $x\geq x_3$, then $f(x)\geq 0$, and when $x\leq x_1$ or $x_2\leq x\leq x_3$, then $f(x)\leq 0$.

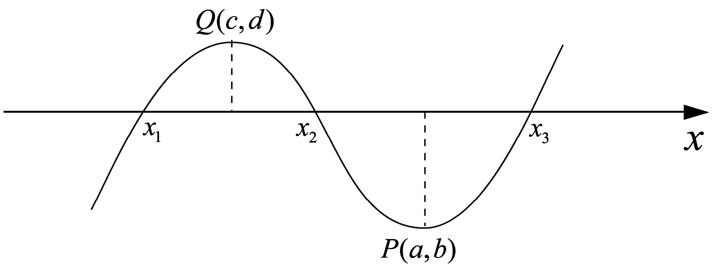

Figure 3

(b) Extrema of a polynomial

By finding the maximum and minimum point of a polynomial, we are able to show $f(x)\geq k_1$ or $f(x)\leq k_2$ for x in some specified intervals.

This can be visualized by considering the local minimum $P\ (a,b)$ and local maximum $Q\ (c,d)$ in Figure 3.

Example 8

Solve $\left|x^2-3\right| < \left|x+1\right|$.

[Solution]

Squaring both sides,

$$(x^2-3)^2 < (x+1)^2.$$

Solving, $(x^2-3)^2-(x+1)^2<0$

$$[(x^2-3)+(x+1)][(x^2-3)-(x+1)]<0$$

$$(x^2+x-2)(x^2-x-4)<0\cdots\cdots(*)$$

The roots of the biquadratic equation

$$(x^2+x-2)(x^2-x-4)=0$$

are found by solving two quadratic equations,

$$x^2+x-2=0, \quad x^2-x-4=0$$

$$x=-2,\ 1, \quad x=\frac{1\pm\sqrt{17}}{2}.$$

Therefore, the graph of $f(x)=(x^2+x-2)(x^2-x-4)$ is sketched as follows

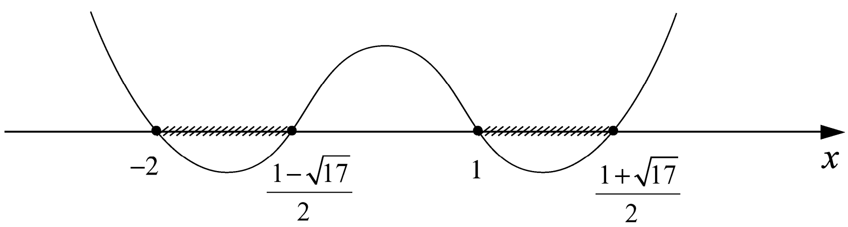

Figure 4

Observe that inequality (*) is satisfied

when $-2<x<\dfrac{1-\sqrt{17}}{2}$ or $1<x<\dfrac{1+\sqrt{17}}{2}$.

Example 9

Show that $x^4-8x^3+22x^2-24x+8\le 0$ when $1\le x\le 3$.

[Solution]

Let $y = f(x) = x^4 - 8x^3 + 22x^2 - 24x + 8$.

Differentiating $f(x)$,

$$f'(x) = 4x^3 - 24x^2 + 44x - 24$$
$$= 4(x^3 - 6x^2 + 11x - 6).$$

By factorization,

$$x^3 - 6x^2 + 11x - 6 = (x-1)(x^2 - 5x + 6)$$
$$= (x-1)(x-2)(x-3).$$

Putting $f'(x) = 0$, we obtain the turning points $x = 1, 2, 3$.

Correspondingly, $y = -1, 0, -1$.

$$f''(x) = 4(3x^2 - 12x + 11)$$

Then $f''(1) = 4(3 - 12 + 11) > 0 \quad \Rightarrow$ local minimum at $x = 1$.

$f''(2) = 4(3\times 4 - 12\times 4 + 11) < 0 \quad \Rightarrow$ local maximum at $x = 2$.

$f''(3) = 4(3\times 9 - 12\times 3 + 11) > 0 \quad \Rightarrow$ local minimum at $x = 3$.

Hence, we can see the graph of $f(x)$ is below the x-axis in $[1,3]$, i.e. $x^4 - 8x^3 + 22x^2 - 24x + 8 \le 0$, when $1 \le x \le 3$.

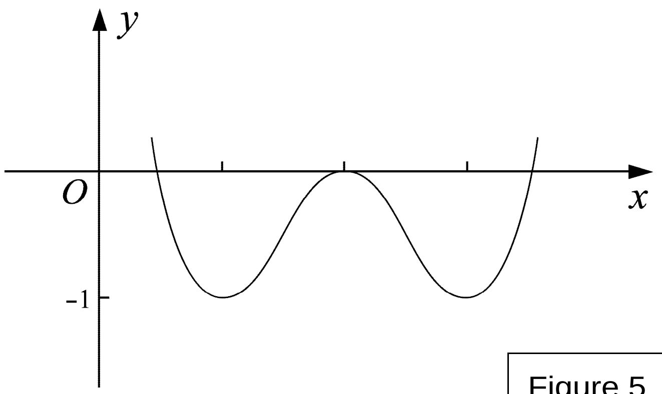

Figure 5

Example 10

Let $x \ge -1$ and $0 < \alpha < 1$.

(a) Prove that $(1+x)^\alpha \le 1 + \alpha x$.

(b) Hence, show that for positive numbers p & q,

$p^{\alpha}q^{1-\alpha} \le \alpha p + (1-\alpha)q$.

[Solution]

(a) Suppose α is rational. Then $0 < \alpha < 1$ indicates $\alpha = \dfrac{p}{q}$ where p & q are positive integers and $p < q$.

Thus, $(1+x)^{\alpha} = (1+x)^{\frac{p}{q}} = \sqrt[q]{(1+x)^{p}}$

$$= \sqrt[q]{\underbrace{(1+x)(1+x)\cdots(1+x)}_{p\text{ terms}} \cdot \underbrace{1\cdot 1\cdots 1}_{(q-p)\text{ terms}}}$$

$$\le \frac{(1+x)+(1+x)+\cdots+(1+x)+1+1\cdots+1}{q} \quad (\because \text{ G.M.} \le \text{A.M.})$$

$$= \frac{p(1+x)+(q-p)}{q} = \frac{px+q}{q}$$

$$= 1 + \frac{p}{q}x = 1 + \alpha x .$$

By the way, the equality holds when $x = 0$.

For the case of α being irrational, we use a sequence of rational numbers $\{r_n\}$ in the interval $(0,1)$, which converges to α ,

i.e. $\lim\limits_{n\to\infty} r_n = \alpha$.

Since $(1+x)^{r_n} \le 1 + r_n x$, $0 < r_n < 1$,

we obtain

$$(1+x)^{\alpha} = \lim_{n\to\infty}(1+x)^{r_n} \le \lim_{n\to\infty}(1+r_n x) = 1 + \alpha x .$$

Obviously, equality holds when $x = 0$.

An alternative method is to use differential calculus.

Let $f(x) = (1+x)^{\alpha} - \alpha x - 1$.

Differentiating and putting the first derivative equal to zero,

$$f'(x) = \alpha(1+x)^{\alpha-1} - \alpha = 0 \quad \Rightarrow \quad x = 0 .$$

Moreover, $f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$, $f''(0) = \alpha(\alpha-1) < 0$ $(\because\ 0 < \alpha < 1)$

$\Rightarrow$ $f(x)$ has a local maximum at $x = 0$.

Hence $f(x) \le f(0)$ $\forall x$

$(1+x)^{\alpha} - \alpha x - 1 \le 0$, $\quad [\because f(0)=0]$

$(1+x)^{\alpha} \le 1+\alpha x$.

(b) Note that $\dfrac{p}{q}$ can be expressed as $1+x$ where $x \ge -1$.

Then $\dfrac{p}{q} = 1+x \;\Rightarrow\; x = \dfrac{p}{q} - 1$.

By (a), $\left(\dfrac{p}{q}\right)^{\alpha} \le 1+\alpha\left(\dfrac{p}{q}-1\right)$.

Multiplying both sides by q,

$$\left(\frac{p}{q}\right)^{\alpha} \cdot q \le q + \alpha p - \alpha q$$

$$p^{\alpha} q^{1-\alpha} \le \alpha p + (1-\alpha) q \,.$$