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F.6 MATHEMATICS MODULE 2

Unit 13 – Matrices and Determinants

(English Version)

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Unit 13 – Matrices and Determinants

A. Fundamental Framework of Matrices

I. Basic concepts and algebraic operations

A **matrix** A of size (order) $m \times n$ is a rectangular array of numbers with m (horizontal) rows and n (vertical) columns,

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ or } \left(a_{ij}\right) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The element a_{ij} (in the i^{th} row and the j^{th} column) is called the $i \times j$ or (i, j) entry of the matrix.

Two matrices A and B are said to be **equal** if they have the same size and the corresponding entries are equal.

If A and B are of the same size $m \times n$, then $A + B$, the **sum** of A and B, is a matrix of size $m \times n$, with its entries equal to the sum of the corresponding entries of A and B.

Let A be a matrix of size $m \times n$, and λ a certain number, then λA is a matrix of size $m \times n$, with its entries equal to λ times the corresponding entries of A. λA is the **scalar multiple** of A by λ.
By definition, $A - B = A + (-1)B$. And $(-1)B$ is usually written as $-B$.

Let $A = [a_{ij}]$ be an $m \times n$ matrix, and $B = [b_{ij}]$ be an $n \times p$ matrix.
Then the i^{th} row of A is $[a_{i1}, a_{i2} \cdots a_{in}]$, and

the j^{th} column of B is $\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$.

The **matrix product** of A and B, written as AB, is a matrix $[c_{ij}]$ of size $m\times p$ defined as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

a_{i1}: 1^{st} element of the i^{th} row of A.

b_{1j}: 1^{st} element of the j^{th} column of B.

For example, taking the 2^{nd} row of A and the 1^{st} column of B, we obtain

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1} .$$

Note:

The definition of matrix multiplication requires that the number of columns of the first factor A be the same as the number of rows of the second factor B in order to form the product AB. If this condition is not satisfied, the product is undefined.

Example 1

Given the matrices,

$$A=\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 4\\ -2 & 5 & -1\end{bmatrix},\quad B=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 2\\ -1 & 2 & 1\end{bmatrix},\quad C=\begin{bmatrix}1\\ -1\\ 2\end{bmatrix},$$

compute the following

$A+B$, $A+C$, $B-3A$, AB, AC, CA.

Solution:

$$A+B=\begin{bmatrix}1+0 & -1+1 & 2+1\\ 0+1 & 3+0 & 4+2\\ -2-1 & 5+2 & -1+1\end{bmatrix}=\begin{bmatrix}1 & 0 & 3\\ 1 & 3 & 6\\ -3 & 7 & 0\end{bmatrix}.$$

Since A is 3×3, C is 3×1, thus $A+C$ is undefined.

$$B-3A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 2\\ -1 & 2 & 1\end{bmatrix}-\begin{bmatrix}3 & -3 & 6\\ 0 & 9 & 12\\ -6 & 15 & -3\end{bmatrix}$$

$$=\begin{bmatrix}0-3 & 1-(-3) & 1-6\\ 1-0 & 0-9 & 2-12\\ -1-(-6) & 2-15 & 1-(-3)\end{bmatrix}=\begin{bmatrix}-3 & 4 & -5\\ 1 & -9 & -10\\ 5 & -13 & 4\end{bmatrix}.$$

$$AB=\begin{bmatrix}1\times 0+(-1)\times 1+2\times(-1) & 1\times 1+(-1)\times 0+2\times 2 & 1\times 1+(-1)\times 2+2\times 1\\ 0\times 0+3\times 1+4\times(-1) & 0\times 1+3\times 0+4\times 2 & 0\times 1+3\times 2+4\times 1\\ (-2)\times 0+5\times 1+(-1)\times(-1) & (-2)\times 1+5\times 0+(-1)\times 2 & (-2)\times 1+5\times 2+(-1)\times 1\end{bmatrix}$$

$$=\begin{bmatrix}-3 & 5 & 1\\ -1 & 8 & 10\\ 6 & -4 & 7\end{bmatrix}.$$

$$AC=\begin{bmatrix}1\times 1+(-1)\times(-1)+2\times 2\\ 0\times 1+3\times(-1)+4\times 2\\ (-2)\times 1+5\times(-1)+(-1)\times 2\end{bmatrix}=\begin{bmatrix}6\\ 5\\ -9\end{bmatrix}.$$

C is 3×1, A is 3×3, and $1\neq 3$, hence CA is undefined.

Special Matrices

Row Matrix: is a matrix with only 1 row, i.e. a $1\times n$ matrix. e.g. $\begin{pmatrix}2 & 5\end{pmatrix}, \begin{pmatrix}3 & 4 & 7\end{pmatrix}$.

Column Matrix: is a matrix with only 1 column, i.e. a $n\times 1$ matrix. e.g. $\begin{pmatrix}2\\6\end{pmatrix}, \begin{pmatrix}4\\5\\8\end{pmatrix}$.

Zero Matrix (or null matrix): is a matrix with all its entries equal zero

e.g. $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$

The zero matrix of order $m\times n$ is denoted by $\mathbf{0}_{m\times n}$ or simply **0**.

Square Matrix: is a matrix with equal number of rows and columns. A $n\times n$ square matrix is said to be of order *n.*

e.g. $\begin{pmatrix}2 & 2\\4 & 7\end{pmatrix}, \begin{pmatrix}0 & 2 & 5\\7 & 1 & -9\\2 & 4 & 3\end{pmatrix}$ are square matrices of order 2 and 3 respectively.

Diagonal Matrix: is a square matrix A with all its entries a_{ij}, where $i\neq j$, equal to zero.

e.g. $\begin{pmatrix}2 & 0 & 0\\0 & -3 & 0\\0 & 0 & 4\end{pmatrix}$

Identity Matrix or Unit Matrix: is a diagonal matrix A with all its entries $a_{ii}=1$. A unit matrix of order *n* is denoted by I_n or simply *I*.

e.g. $I_1=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, I_3=\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$

General properties of matrices

Let A, B, and C be matrices, and λ, λ_1 & λ_2 be scalars. Provided the operations make sense, then

1. $A+B=B+A$ (Commutative law of addition)

2. $A+(B+C)=(A+B)+C$ (Associative law of addition)

3. Let 0 be a zero matrix, then $A+0=0+A=A$ (Additive identity)

4. For any matrix A, $A+(-1)A=A+(-A)=0$, a zero matrix
(Additive inverse)

5. $\lambda_1(\lambda_2 A)=\lambda_1\lambda_2(A)$

6. $(\lambda_1+\lambda_2)A=\lambda_1 A+\lambda_2 A$

7. $\lambda(A+B)=\lambda A+\lambda B$

8. $A(BC)=(AB)C$ (Associative law of multiplication)

*9. $(A+B)C=AC+BC$
(Distributive law: Multiplication is right distributive over addition)

*10. $A(B+C)=AB+AC$
(Distributive law: Multiplication is left distributive over addition)

11. $(\lambda A)B=\lambda(AB)$

12. $A(\lambda B)=\lambda(AB)$

13. Let I be an identity matrix. Then $AI=A$ and $IA=A$.

*14. The transpose of a matrix, A^T

e.g. $A=\begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & -2 \end{bmatrix}$ $A^T=\begin{bmatrix} 2 & -1 \\ 3 & 2 \\ 4 & -2 \end{bmatrix}$, $B=\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $B^T=\begin{bmatrix} 1 & 3 \end{bmatrix}$.

*15. $(AB)^T=B^T A^T$

Example 2

Find $\begin{bmatrix} a & b \\ b & a \end{bmatrix}^n$.

Solution

Let $P=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Obviously $P^2=I$. Also $\begin{bmatrix} a & b \\ b & a \end{bmatrix}=aI+bP$.

Then by the Binomial theorem,

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}^n=(aI+bP)^n=a^nI+C_1^na^{n-1}bP+C_2^na^{n-2}b^2P^2+C_3^na^{n-3}b^3P^3+\cdots+b^nP^n$$

$$=(a^n+C_2^na^{n-2}b^2+C_4^na^{n-4}b^4+\cdots)I+(C_1^na^{n-1}b+C_3^na^{n-3}b^3+\cdots)P$$

$$=\frac{1}{2}\left[(a^n+C_1^na^{n-1}b+C_2^na^{n-2}b^2+\cdots)+(a^n-C_1^na^{n-1}b+C_2^na^{n-2}b^2-\cdots)\right]I$$

$$+\frac{1}{2}\left[(a^n+C_1^na^{n-1}b+C_2^na^{n-2}b^2+\cdots)-(a^n-C_1^na^{n-1}b+C_2^na^{n-2}b^2-\cdots)\right]P$$

$$=\left[\frac{(a+b)^n+(a-b)^n}{2}\right]I+\left[\frac{(a+b)^n-(a-b)^n}{2}\right]P$$

$$=\frac{1}{2}\begin{bmatrix} (a+b)^n+(a-b)^n & (a+b)^n-(a-b)^n \\ (a+b)^n-(a-b)^n & (a+b)^n+(a-b)^n \end{bmatrix}.$$

Note: Another method of finding $\begin{bmatrix} a & b \\ b & a \end{bmatrix}^n$ is to make use of a matrix $P=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and its inverse P^{-1}. It is given in Example 5, Section C of this Chapter.

Instant Drill 1.

1. Evaluate the following expressions:

a) $2(1 \quad 4)+3(5 \quad 2)$

b) $-\begin{pmatrix} 3 & -2 \\ 6 & 0 \end{pmatrix}-4\begin{pmatrix} -2 & 0 \\ 1 & 5 \end{pmatrix}$

c) $3\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}-6\begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}+2\begin{pmatrix} 8 \\ 5 \\ -17 \end{pmatrix}$

d) $2\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}-3\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 1 \\ -1 & 0 & 0 \end{pmatrix}$

2. Let $P=\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & -3 \\ 4 & 0 & 1 \end{pmatrix}$, $Q=\begin{pmatrix} -1 & 0 & 1 \\ -2 & -2 & 0 \\ 0 & 8 & 1 \end{pmatrix}$ and $R=\begin{pmatrix} -4 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 2 & -4 \end{pmatrix}$. Evaluate each of the following

a) $3P-4Q+5R$

b) $R+P+3Q-4R-2P+2Q+3R$

c) $2P-Q+4R-2Q-7R+P$

Exercise 1.

1. It is given that $A=\begin{pmatrix}2&5\\3&1\end{pmatrix}$ and $B=\begin{pmatrix}a&2b\\c+1&\frac{d}{3}\end{pmatrix}$. If $A=B$, find the values of a, b, c and d.

2. It is given that $A=\begin{pmatrix}4&x\\0&4\\z&y\end{pmatrix}$ and $B=\begin{pmatrix}a+2&3a\\b&2c\\ac&b-c\end{pmatrix}$. If $2A=B$, find the values of a, b, c, x, y and z

3. Evaluate $\begin{pmatrix}2&-1&4\\5&0&-3\end{pmatrix}\begin{pmatrix}0&-1\\2&5\\1&-4\end{pmatrix}$

4. Let $\begin{pmatrix}2&0&4\\1&-1&t\end{pmatrix}\begin{pmatrix}3\\0\\5\end{pmatrix}=\begin{pmatrix}k\\13\end{pmatrix}$. Find the values of k and t

5. Let $\begin{pmatrix}0\\s\\-3\end{pmatrix}\begin{pmatrix}-2&1&t\end{pmatrix}=\begin{pmatrix}0&0&0\\8&-4&32\\6&-3&24\end{pmatrix}$. Find the values of s and t

6. Let $\begin{pmatrix}3&-2\\4&x\end{pmatrix}\begin{pmatrix}7&-2\\5&1\end{pmatrix}=\begin{pmatrix}11&-8\\y&y\end{pmatrix}$. Find the values of x and y

7. Let $A=\begin{pmatrix}3x&y\\-2z&0\end{pmatrix}$. If $A\begin{pmatrix}4\\1\end{pmatrix}=\begin{pmatrix}27\\-4\end{pmatrix}$ and $A\begin{pmatrix}-1\\3\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$, find A

B. Determinants

I. Basic Concepts

Let $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then the **determinant** of A, denoted by $|A|$, $\det(A)$, or $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ equals, by definition, $a_{11}a_{22}-a_{12}a_{21}$.

The determinant of an 3×3 matrix is defined in terms of determinants of 2×2 matrices. Thus let

$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then the determinant of A is given by

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

The formula for defining the determinant of a larger size square matrix, in terms of smaller size matrices, is given as follows

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{cases} \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij} \text{ for any } i \cdots\cdots (a) \\ \sum_{i=1}^{n} a_{ij}(-1)^{i+j}M_{ij} \text{ for any } j \cdots\cdots (b) \end{cases}$$

where M_{ij} denotes the determinant of the submatrix obtained from a square matrix $A=\left[a_{ij}\right]_{(n\times n)}$ by deleting the i^{th} row and the j^{th} column of A (yielding a square matrix of order $n-1$).

$C_{ij}=(-1)^{i+j}M_{ij}$ is called the **cofactor** of a_{ij},

and M_{ij} is called the **minor** of a_{ij}.

(a) & (b) are called, respectively, the cofactor (minor) expansion of $\det(A)$ by the i^{th} row and j^{th} column. It follows that

(i) $\det(A^T)=\det(A)$

(ii) $\det(A)=0$ if A has a zero row or zero column.

The proof is quite simple and is left to the student.

Example 3

Let $A=\begin{bmatrix}1 & 2 & 3\\ -1 & 3 & 1\\ 1 & 3 & 2\end{bmatrix}$. Find the minors and cofactors of A and evaluate $|A|$.

Solution:

$M_{11}=\begin{vmatrix}3 & 1\\ 3 & 2\end{vmatrix}=3$, $M_{12}=\begin{vmatrix}-1 & 1\\ 1 & 2\end{vmatrix}=-3$, $M_{13}=\begin{vmatrix}-1 & 3\\ 1 & 3\end{vmatrix}=-6$,

$M_{21}=\begin{vmatrix}2 & 3\\ 3 & 2\end{vmatrix}=-5$, $M_{22}=\begin{vmatrix}1 & 3\\ 1 & 2\end{vmatrix}=-1$, $M_{23}=\begin{vmatrix}1 & 2\\ 1 & 3\end{vmatrix}=1$

$M_{31}=\begin{vmatrix}2 & 3\\ 3 & 1\end{vmatrix}=-7$, $M_{32}=\begin{vmatrix}1 & 3\\ -1 & 1\end{vmatrix}=4$, $M_{33}=\begin{vmatrix}1 & 2\\ -1 & 3\end{vmatrix}=5$.

Hence $C_{11}=3$, $C_{12}=3$, $C_{13}=-6$, $C_{21}=5$, $C_{22}=-1$, $C_{23}=-1$,
$C_{31}=-7$, $C_{32}=-4$, $C_{33}=5$.

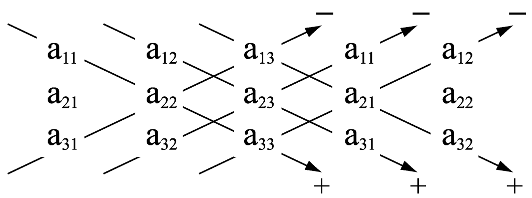
We can use cofactor expansion by any row or column, e.g. $|A|=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}$ (expansion by the 1^{st} row)

$=(1)(3)+(2)(3)+(3)(-6)=-9$,

or $a_{12}C_{12}+a_{22}C_{22}+a_{32}C_{32}$ (expansion by the 2^{nd} column)

$=(2)(3)+(3)(-1)+(3)(-4)=-9$.

Note: To evaluate a 3×3 determinant, we can use the **Rule of Sarrus.**



$$\det(A)=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{31}a_{22}a_{13}-a_{32}a_{23}a_{11}-a_{33}a_{21}a_{12}.$$

Instant Drill 2.

1. Consider $\begin{vmatrix} 2 & 4 & 1 \\ 3 & 1 & 5 \\ 0 & 1 & 2 \end{vmatrix}$.

(a) Find M_{23}.

(b) Find A_{31}.

(c) Find A_{32}.

2. Find the value of each of the following determinants

a. $\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}$

b. $\begin{vmatrix} 2 & 0 \\ 7 & -9 \end{vmatrix}$

c. $\begin{vmatrix} 8 & -5 \\ -6 & 9 \end{vmatrix}$

d. $\begin{vmatrix} -1 & -9 \\ -6 & -4 \end{vmatrix}$

3. Find the value of each of the following determinants by cofactor expansion.

a. $\begin{vmatrix} 0 & 2 & 5 \\ 1 & 4 & 8 \\ 0 & 9 & 7 \end{vmatrix}$

b. $\begin{vmatrix} 9 & 0 & 4 \\ -3 & 5 & 0 \\ 7 & 0 & 2 \end{vmatrix}$

c. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ -4 & 5 & 7 \end{vmatrix}$

d. $\begin{vmatrix} 3 & 0 & -2 \\ 1 & 5 & 1 \\ 4 & 2 & 7 \end{vmatrix}$

e. $\begin{vmatrix} 1 & 3 & 5 \\ 2 & -4 & 3 \\ -3 & 11 & -1 \end{vmatrix}$

f. $\begin{vmatrix} 3 & -6 & 5 \\ 1 & -2 & -1 \\ 4 & 8 & 7 \end{vmatrix}$

II. Further properties of determinants

Aside from using cofactor expansion to evaluate determinants, the following properties are often helpful in computation.

1. Let A be of the form $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & a_{nn} \end{vmatrix}$, i.e. entries below the diagonal are zeros.

   Then $\det A = |A| = a_{11}a_{22}\cdots a_{nn}$.

2. Let A be an $n \times n$ matrix. And B is obtained by interchanging any two rows (or columns) of A. Then $|A| = -|B|$.

3. If one row of A is a multiple of another row, then $|A| = 0$.

4. If B is the matrix obtained by adding a multiple of one row (or column) of A to another row (column), then $|B| = |A|$.

*5. If B is the matrix obtained by multiplying each entry of a row (or column) of A by the same number k, then $|B| = k|A|$.

*6. The determinant of the product of two square matrices of order n is the product of their determinants. That is, $|AB| = |A||B|$.

Example 4

Simplify the determinant $\begin{vmatrix} a+b & ab & a^2b^2 \\ b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \end{vmatrix}$, assuming a, b & c are nonzero.

Solution:

$$\begin{vmatrix} a+b & ab & a^2b^2 \\ b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \end{vmatrix} = \begin{vmatrix} a+b-(b+c) & ab-bc & a^2b^2-b^2c^2 \\ b+c & bc & b^2c^2 \\ c+a-(b+c) & ca-bc & c^2a^2-b^2c^2 \end{vmatrix} \quad \begin{matrix} (R1-R2) \\ \\ (R3-R2) \end{matrix}$$

$$= \begin{vmatrix} a-c & b(a-c) & b^2(a+c)(a-c) \\ b+c & bc & b^2c^2 \\ a-b & c(a-b) & c^2(a+b)(a-b) \end{vmatrix}$$

$$= (a-c)(a-b)\begin{vmatrix} 1 & b & b^2(a+c) \\ b+c & bc & b^2c^2 \\ 1 & c & c^2(a+b) \end{vmatrix}$$

$$= \frac{(a-c)(a-b)}{bc}\begin{vmatrix} 1\cdot c-(b+c) & b\cdot c-bc & b^2(a+c)\cdot c-b^2c^2 \\ b+c & bc & b^2c^2 \\ 1\cdot b-(b+c) & c\cdot b-bc & c^2(a+b)\cdot b-b^2c^2 \end{vmatrix}$$

$$= \frac{(a-c)(a-b)}{bc}\begin{vmatrix} -b & 0 & b^2ac \\ b+c & bc & b^2c^2 \\ -c & 0 & c^2ab \end{vmatrix} = (a-c)(a-b)\begin{vmatrix} -1 & 0 & bac \\ b+c & bc & b^2c^2 \\ -1 & 0 & cab \end{vmatrix}$$

$$= (a-c)(a-b)\cdot bc\begin{vmatrix} -1 & bac \\ -1 & cab \end{vmatrix}$$

$$= bc(a-b)(a-c)[-cab-(-bac)] = 0.$$

Instant Drill 3.

1.Prove the following identities.

a. $\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = (a+2)(a-1)^2$

b. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x)$

c. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2-bc & b^2-ac & c^2-ab \end{vmatrix} = 0$

d. $\begin{vmatrix} a & b & c \\ b-c & a+c & b-a \\ b+c & a-c & b+a \end{vmatrix} = 2(a+b)(a-c)(a-b+c)$

2. Factorize the determinant

$\begin{vmatrix} x^3 & y^3 & z^3 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$.

Exercise 2.

1.Factorize $\begin{vmatrix} 4 & x+1 & x+1 \\ x+1 & (x+2)^2 & 1 \\ x+1 & 1 & (x+2)^2 \end{vmatrix}$.

2.Show that $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$.

3.(a) Factorize $\begin{vmatrix} x & 2 & -5 \\ 2 & x & -5 \\ -5 & 2 & x \end{vmatrix}$.

(b) Hence solve $\begin{vmatrix} x & 2 & -5 \\ 2 & x & -5 \\ -5 & 2 & x \end{vmatrix} = 0$

C. The Inverse of Matrix

I. Definition

Suppose A is a square matrix of size $n \times n$. If there exists an $n \times n$ matrix B such that $AB = BA = I$ (the $n \times n$ identity matrix), then B is called the **inverse** of A. Obviously, the matrix A is also the inverse of B.

A matrix which has an inverse is said to be **invertible** or **nonsingular**. Otherwise, it is said to be **singular** or **non-invertible**.

We denote the inverse of A be A^{-1}

FORMULA

$$A^{-1} = \frac{1}{|A|}[C_{ij}]_{n \times n}^{T}$$

Example 5

(a)Show that $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{b} \\ \frac{1}{a} & 0 \end{pmatrix}$, where $a \neq 0$ and $b \neq 0$.

(b) Hence find $\begin{pmatrix} 0 & -2 \\ 3 & 0 \end{pmatrix}^{-1}$.

Solution:

Instant Drill 4.

1.(a)Show that $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$, where k is a real number.

(b)Hence find the inverse of the following matrices.

(i) $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$

2. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where a, b, c and d are real numbers.

(a) Find the values of a, b, c and d.

(b) Hence find $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}^{-1}$.

3. It is given that $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$.

(a) Find $A^2 - 3A + 2I$.

(b) Hence find A^{-1}.

Exercise 3.

1. Let $X=\begin{pmatrix}5&1&0\\0&5&0\\0&1&5\end{pmatrix}$.

   (a) Show that $(X-4I)(6I-X)=I$.

   (b) Show that $X^2-10X+25I=\mathbf{0}$.

   (c) Hence find X^{-1}.

2. Let $A=\begin{pmatrix}1&-2\\3&1\end{pmatrix}$.

   (a) If $A^2+xA+yI=\mathbf{0}$, find the values of x and y.

   (b) Hence find A^{-1}.

3. It is given that $A=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$.

   (a) Find A^2.

   (b) Hence find A^{-1}.

   (c) Find A^{101}.

4. Let $A=\begin{pmatrix}2&-3\\1&1\end{pmatrix}$ and $B=\begin{pmatrix}4&3\\-1&5\end{pmatrix}$.

   (a) Prove that $AB=BA$.

   (b) Verify that $(A+B)^2=A^2+2AB+B^2$.

5. Let $A=\begin{pmatrix}0&1\\2&-1\end{pmatrix}$ and $B=\begin{pmatrix}3&0\\1&2\end{pmatrix}$.

   (a) Find AB and BA.

   (b) Find A^2 and B^2.

   (c) Verify that $(A+B)^2\neq A^2+2AB+B^2$.

   (d) Let $C=\begin{pmatrix}1&2\\4&-1\end{pmatrix}$. Verify that $(A+C)^2=A^2+2AC+C^2$.

6. Let $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ and $B=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$.

(a) Show that $AB=BA=I$.

(b) Hence find A^{-1} and B^{-1}.

(c) Prove, by mathematical induction, that for all positive integers n, $A^n=\begin{pmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{pmatrix}$.

(d) Find $(B^n)^{-1}$, where n is a positive integer.

7. Let $A=\begin{pmatrix}4 & 9\\ -1 & -2\end{pmatrix}$.

(a) Prove that $A^2-2A+I=\mathbf{0}$.

(b) Prove, by mathematical induction, that for all positive integers n, $A^n=nA-(n-1)I$.

(c) Find $\begin{pmatrix}4 & 9\\ -1 & -2\end{pmatrix}^{2\,010}$.

8. Let $A=\begin{pmatrix}0 & a & b\\ 0 & 0 & c\\ 0 & 0 & 0\end{pmatrix}$, where a, b and c are real numbers.

(a) Find A^2 and A^3.

(b) Show that $(A^2-A+I)(A+I)=A^3+I$.

(c) Hence find $\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}^{-1}$.

9. Let $P=\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}$.

(a) Prove that $P^3=-I$, and hence prove that $P^6=I$.

(b) Find P^{-1}.

(c) Prove that $P^5+P^4+P^3+P^2+P+I=\mathbf{0}$.

(d) Prove that $(P^{-1})^5+(P^{-1})^4+(P^{-1})^3+(P^{-1})^2+P^{-1}+I=\mathbf{0}$.

10. Let $U=\frac{1}{3}\begin{pmatrix}2 & -2 & 1\\ 2 & 1 & -2\\ 1 & 2 & 2\end{pmatrix}$ and $A=\frac{1}{9}\begin{pmatrix}17 & -8 & 4\\ -8 & 17 & -4\\ 4 & -4 & 11\end{pmatrix}$.

(a) Prove that $U^{-1}=\frac{1}{3}\begin{pmatrix}2 & 2 & 1\\ -2 & 1 & 2\\ 1 & -2 & 2\end{pmatrix}$.

(b) Let $M=UAU^{-1}$. Find M.

(c) It is given that $\begin{pmatrix}c_1 & 0 & 0\\ 0 & c_2 & 0\\ 0 & 0 & c_3\end{pmatrix}^{-1}=\begin{pmatrix}\frac{1}{c_1} & 0 & 0\\ 0 & \frac{1}{c_2} & 0\\ 0 & 0 & \frac{1}{c_3}\end{pmatrix}$ and $\begin{pmatrix}c_1 & 0 & 0\\ 0 & c_2 & 0\\ 0 & 0 & c_3\end{pmatrix}^{n}=\begin{pmatrix}c_1{}^n & 0 & 0\\ 0 & c_2{}^n & 0\\ 0 & 0 & c_3{}^n\end{pmatrix}$ for all positive integers n, where c_1, c_2 and c_3 are real numbers. Find $(A^{-1})^n$.

11. Let $A=\begin{pmatrix}2 & 1 & 0\\ -1 & 5 & 4\end{pmatrix}$. Find A^TA and AA^T.

12. Let $X=\begin{pmatrix}1\\ 2\\ -4\end{pmatrix}$. Find X^TX and XX^T.

13. It is given that A, B and C are square matrices, where $A^2B^T=C$, $|A|=2$, $|C|=3$. Find the value of $|B|$.

14. It is given that P, Q and R are 2×2 matrices, where R is invertible, $P^TQ^3=-R^{-1}$, $P^2R=I$ and $|P|=8$. Find the values of $|Q|$ and $|R|$.

15. Let $M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, where a, b, c and d are real numbers. It is given that $M\begin{pmatrix}-1\\ -2\end{pmatrix}=\begin{pmatrix}-5\\ -9\end{pmatrix}$ and $M\begin{pmatrix}-3\\ -5\end{pmatrix}=\begin{pmatrix}-14\\ -25\end{pmatrix}$.

(a)Find the values of a, b, c and d.

(b) Find M^{-1}.

(c)Find $M^{-1}\begin{pmatrix}-5 & -14\\ -9 & -25\end{pmatrix}$.

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